Some Practical Matching Conditions for Robust Control System Design

J.F. Whidborne
Department of Mechanical Engineering
King’s College London
Strand, London WC2R 2LS, UK
email: james.whidborne@kcl.ac.uk

T. Ishihara, H. Inooka, T. Ono
Graduate School of Information Sciences
Tohoku University
Aoba-yama 01, Aoba-ku
Sendai 980-8579, JAPAN
email: ishiharaiinookaono@control.is.tohoku.ac.jp

T. Satoh
Department of Machine Intelligence and System Engineering
Faculty of Science and Technology
Akita Prefectural University
84-4, Ebinokuchi, Tsuchiya, Honjoh
Akita 015-0055, JAPAN
email: tsatoh@akita-pu.ac.jp

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J.F. Whidborne T. Ishihara H. Inooka T. Ono T. Satoh

Abstract

The principle of matching is a general concept of control system design that gives due regard for the environment in which the system is to operate. That is, given a set of permissible environments and a set of permissible systems, the task of the designer is to choose one environment and one system such that the two are well-matched. For the practical design of most engineering control systems, it is important to explicitly consider uncertainty within the system plant. Therefore, to design the control system within the framework of matching, practical robust matching conditions are required. This report provides sufficient conditions for a robust match for systems with plants modelled as having additive, multiplicative or factored uncertainty. Both upper and lower bounds on the sufficient conditions for a match are presented.

1 Introduction

The principle of matching [Zak91] gives due regard for the environment in which the system is to operate. That is, given a set of admissible environments and a set of admissible systems, the task of the designer is to choose one environment and one system such that the two are well-matched. Suppose that an input \( w \) from the environment to the system belongs to a set of possible inputs, \( \mathcal{P} \). This set characterizes the environment. If the response of the system to the input is tolerable in some well-defined sense, then the input is described as tolerable. The set of all tolerable inputs [Zak89], \( \mathcal{T} \), characterize the system. The system and the environment are said to be matched if the possible set, \( \mathcal{P} \), is a subset of the tolerable set, \( \mathcal{T} \) (i.e. \( \mathcal{P} \subseteq \mathcal{T} \)).

Practical conditions for the system to be matched to the environment have been established for several classes of systems. For example, [Zak87, Zak91, Rut94a, Rut94b, Zak96] present matching conditions for continuous time linear systems. In [SII99], matching conditions on the Youla parameter are presented. In [SII97], matching conditions for discrete time linear systems are presented, and in [Whi92] and [OII00], matching conditions for linear sampled data feedback systems are established.

However, for the practical design of most engineering control systems, it is important to explicitly consider uncertainty within the system, thus ensure the robustness of the design. Thus matching conditions which guarantee the system is matched to the environment despite the system uncertainty are required; this is termed a robust match. Some practical conditions for a class of uncertainty have been obtained [Zak83, Zak96], however, this class is restricted and can be interpreted as a class of plants with stable additive uncertainty.

This report extends and clarifies results in [WZII00]. By means of a functional analytical approach, practical sufficient conditions for a robust match are obtained for a standard feedback system with (i) plants with stable additive uncertainty, (ii) plants with stable multiplicative
uncertainty and (iii) plants with additive uncertainty on its co-prime factors. In addition, lower bounds on the conditions are obtained.

Some preliminary definitions are presented in the next section. The principle of matching is reviewed in Section 3. In Section 4, a standard feedback problem is posed. In Section 5, conditions for a robust match for certain models of uncertainty are developed. Finally, there is some discussion and conclusions are drawn.

2 Preliminaries

Definition 1. For a signal $w(t)$, define the $L_\infty$-norm as

$$\|w\|_\infty := \sup_{t \in \mathbb{R}} |w(t)|$$

In the sequel, it is assumed that the systems under consideration are single-input single-output, causal, linear time varying systems. Such systems can be described by the generalized impulse response function $v(t, \tau, \delta)$, that is the output at time $t$ to a generalized unit impulse function $\delta(t - \tau)$ applied at time $\tau$ when the system is initially in the equilibrium state. The output $x(t, w)$ to an input $w(t)$, $w(t) = 0$ for $t < 0$, is given by

$$x(t, w) = (v(\delta) * w)(t) := \int_0^t v(t, \tau \delta)w(\tau)d\tau.$$  

(2)

Definition 2. For the causal linear time varying system, $v$, with a generalized response $v(t, \tau, \delta)$ to an generalized impulse function $\delta(\tau - t)$ applied at time $\tau$ when the system is initially in the equilibrium state. The output $x(t, w)$ to an input $w(t)$, $w(t) = 0$ for $t < 0$, is given by

$$x(t) = \left(v(\delta) * \text{d}w/dt\right)(t) := \int_0^t v(t, \tau \delta)\text{d}w(\tau)d\tau.$$  

(6)

where $\text{d}w/dt$.

The assumption here is that the generalized function may include generalized functions (distributions) such as steps and impulses which are continuously differentiable. For a more rigorous treatment of such system descriptions, see, for example, [Vid93]

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Proof. From (2)

\[ v(t, \tau, h) = \int_\tau^t v(t, \tau', \delta) d\tau' \]  

which can be expressed as

\[ v(t, \tau, h) = \int_0^t v(t, \tau', \delta) d\tau' - \int_0^\tau v(t, \tau', \delta) d\tau' \]  

\[ = v(t, 0, h) - \int_0^\tau v(t, \tau', \delta) d\tau' \]  

and differentiating gives

\[ \frac{dv(t, \tau, h)}{d\tau} = -v(t, \tau, \delta) \]  

Integrating (2) by parts and substituting (10) gives

\[ v(t, v) = -v(t, \tau, h)w(\tau) \bigg|_0^t + \int_0^t v(t, \tau, h)w(\tau) d\tau \]  

and since \( w(0) = 0 \) and \( v(t, t, h) = 0 \), (6) is obtained.

3 A framework for matching

Assume that the system input \( w : \mathbb{R} \to \mathbb{R} \) belongs to the set \( \mathcal{P} \) which includes the set of all possible inputs to the system. This set characterizes the environment.

Let the response of the signals and states of interest of the system be \( x(w) = (x_1(w), \ldots, x_m(w)) \). Let it be assumed that the purpose of the control is to ensure that the maximum absolute value of each response stays within some prescribed bound for all time. This is a requirement for most practical control systems. This requirement is expressed as

\[ \| x_i(w) \|_\infty \leq \varepsilon_i, \ i = 1, \ldots, m. \]  

The set of tolerable inputs \( \mathcal{T} \) is defined as

\[ \mathcal{T} := \{ w : \| r_i(w) \|_\infty \leq \varepsilon_i, \ i = 1, \ldots, m \}. \]  

According to the principle of matching [Zak91], the aim of the design is to choose the control system and the environment such that the tolerable input space, \( \mathcal{T} \), includes the possible input space \( \mathcal{P} \), that is \( \mathcal{P} \subseteq \mathcal{T} \). When this condition is satisfied, the system and the environment are said to be matched.

If the possible set of inputs, \( \mathcal{P} \), is modelled as a set of inputs \( \mathcal{W} \), that is \( \mathcal{W} \subseteq \mathcal{P} \), then clearly if \( \mathcal{W} \subseteq \mathcal{T} \), then the system and the environment are matched. For this report, two input sets are considered. The set \( \mathcal{W}_0 \) essentially consists of all inputs with an upper bound on the input magnitude. The set \( \mathcal{W}_1 \) essentially consists of all inputs with an upper bound on the first derivative of the input.

**Definition 3.** The set \( \mathcal{W}_0 \) is defined as

\[ \mathcal{W}_0 := \{ w : w(t) = 0 \text{ for } t \leq 0, \ |w|_\infty \leq \mu_0 \}. \]
This is the set of all inputs bounded in magnitude, and is one of the simplest models of the input set. This model is used in this paper for completeness, it is generally not the most appropriate model [Zak86, WL93].

**Definition 4.** The set $\mathcal{W}_1$ is defined as

$$\mathcal{W}_1 := \{ w : w(t) = 0 \text{ for } t \leq 0, \| \dot{w} \|_\infty \leq \mu_1 \}.$$  (15)

This is the set of all inputs bounded in their rate of change. It is a more practical model than that described by $\mathcal{W}_0$ [Zak86, WL93]. For examples of other models, see [Zak96, Rut92, WL93].

**Lemma 1.** For a linear time invariant system, $v$, with input $w \in \mathcal{W}_0$ and output $x(t)$ then

$$\sup_{w \in \mathcal{W}_0} \| x \|_\infty = \| v(\delta) \|_1 \mu_0.$$  (16)

This is well-known (for example, see [Hor63]). The result can be extended for time varying systems [Wil70, p. 108] so that, for a system, $v$, with input $w(t)$ and output $x(t)$ that

$$\sup_{w \in \mathcal{W}_0} \| x \|_\infty = \sup_{\tau} \| v(\delta) \|_1 \mu_0.$$  (17)

**Lemma 2.** For a linear time invariant system, $v$, with input $w \in \mathcal{W}_1$ and output $x(t)$ then

$$\sup_{w \in \mathcal{W}_1} \| x \|_\infty = \| v(h) \|_1 \mu_1.$$  (18)

This relationship is less well-known ([Zak86]). The above two lemmas provide a means of obtaining the tolerable input space, $\mathcal{T}$, for linear systems without uncertainty [Zak91]. They will be used in the sequel.

## 4 A standard feedback problem

![Figure 1: A standard feedback problem](image)

Consider the standard SISO system shown in Figure 1. If internal stability is assumed, the error is given by

$$e = w - g \ast u$$  (19)

and the control is

$$u = k \ast e$$  (20)
and where the plant is represented by the piece-wise continuous function $g : \mathbb{R}_+ \to \mathbb{R}$ and the controller by the piece-wise continuous function $k : \mathbb{R}_+ \to \mathbb{R}$. It is assumed in the sequel that the controller is a linear time invariant system.

In general, the plant is not known exactly, but may be modelled as belonging to some known set $\mathcal{G}$, that is $g \in \mathcal{G}$. Assume that the sole signal of interest is the error, $e$. The requirement on the design is then

$$\| e(w, g) \|_\infty \leq \varepsilon,$$

and the set of tolerable inputs $\mathcal{T}$ is

$$\mathcal{T} = \left\{ w : \sup_{g \in \mathcal{G}} \| e(w) \|_\infty \leq \varepsilon \right\}.$$  

A sufficient practical condition for a robust match between this system and the environment is given by

$$\hat{e} \leq \varepsilon$$

where

$$\hat{e} := \sup_{w \in \mathcal{W}} \sup_{g \in \mathcal{G}} \| e(w) \|_\infty$$

Clearly, $\hat{e} \leq \varepsilon \Rightarrow \mathcal{P} \subseteq \mathcal{T}$, and is a sufficient condition for a robust match.

In the following sections, $\hat{e}$ is calculated for the above system for various descriptions of the uncertainty for both $\mathcal{W}_0$ and $\mathcal{W}_1$. A linear time invariant approximation of the plant, known as the nominal plant, $g_0$, is taken in each instance, and since the system is internally stable, it is assumed that the nominal error response, $e_0(t, w)$, is bounded. So, from Figure 1

$$e_0 = w - g_0 * u_0, \quad u_0 = k * e_0,$$

and hence

$$e_0 = s * w,$$

where

$$s := (1 + g_0 * k)^{-1}.$$

### 5 Matching conditions for uncertain plants

#### 5.1 Additive uncertainty

Let $g \in \mathcal{G}_+$ where $\mathcal{G}_+$ represents some set of plants. Define a nominal linear time invariant plant $g_0$ such that $g_0 \in \mathcal{G}_+$ and

$$g := g_0 + z_+$$

where $z_+$ is termed the additive uncertainty or perturbation. Define

$$\mathcal{G}_+ := \{ g = g_0 + z_+ : \| z_+(\tau, \delta) \|_1 \leq \gamma \ \forall \ \tau \in \mathbb{R} \}.$$
Theorem 1. For the feedback system of Figure 1, if $g \in G_+$ and $w \in W_0$, then

$$\frac{\|s(\delta)\|_{1\mu_0}}{1 + \gamma \| (k * s)(\delta) \|_1} \leq \hat{e} \leq \frac{\|s(\delta)\|_{1\mu_0}}{1 - \gamma \| (k * s)(\delta) \|_1}$$

(provided the nominal response $e_0(t, w)$ is bounded (i.e. the nominal closed loop system is stable) and $\gamma \| (k * s)(\delta) \|_1 < 1$).

Proof. Substituting (28) into (19) gives

$$e = w - z_+ * u - g_0 * u$$

which from (27) gives

$$e = s * w - z_+ * k * s * e$$

which gives

$$\| e \|_\infty \leq \| s * w \|_\infty + \| z_+ * k * s * e \|_\infty$$

by the triangular inequality. From Holder’s inequality

$$\| e \|_\infty \leq \| s(\delta) \|_1 \| w \|_\infty + \| (z_+ * k * s)(\tau, \delta) \|_1 \| e \|_\infty.$$  

From the sub-multiplicative property (4)

$$\| e \|_\infty \leq \| s(\delta) \|_1 \| w \|_\infty + \| z_+ (\tau, \delta) \|_1 \| (k * s)(\delta) \|_1 \| e \|_\infty,$$

and from (29)

$$\| e \|_\infty \leq \| s(\delta) \|_1 \| w \|_\infty + \gamma \| (k * s)(\delta) \|_1 \| e \|_\infty \forall g \in G_+.$$  

So if $\gamma \| (k * s)(\delta) \|_1 < 1$, then

$$\sup_{g \in G_+} \| e \|_\infty \leq \frac{\| s(\delta) \|_1 \| w \|_\infty}{1 - \gamma \| (k * s)(\delta) \|_1}$$

which from Lemma 1 gives

$$\sup_{w \in W_0} \sup_{g \in G_+} \| e \|_\infty \leq \frac{\| s(\delta) \|_{1\mu_0}}{1 - \gamma \| (k * s)(\delta) \|_1}$$

and the right-hand inequality of (30).

Now from (33),

$$e - s * w = z_+ * k * s * e$$

which from Holder’s inequality and the sub-multiplicative property of the $L_1$-norm and (29) gives

$$\| s * w - e \|_\infty \leq \gamma \| (k * s)(\delta) \|_1 \| e \|_\infty \forall g \in G_+.$$
The identity $s \ast w = (s \ast w - e) + e$ provides the inequality
\[
\| s \ast w \|_\infty \leq \| s \ast w - e \|_\infty + \| e \|_\infty
\] (42)
from the triangular inequality. From (41)
\[
\gamma \| (k \ast s)(\delta) \|_1 \| e \|_\infty + \| e \|_\infty \geq \| s \ast w \|_\infty \forall g \in \mathcal{G}_+
\] (43)
and since from Lemma 1, $\| s \ast w \|_\infty = \| s(\delta) \|_1 \mu_0$ for all $w \in \mathcal{W}_0$, then
\[
\sup_{w \in \mathcal{W}_0} \sup_{g \in \mathcal{G}_+} \| e \|_\infty \geq \frac{\| s(\delta) \|_1 \mu_0}{1 + \gamma \| (k \ast s)(\delta) \|_1}
\] (44)
and the left-hand inequality of (30).

**Theorem 2.** For the feedback system of Figure 1, if $g \in \mathcal{G}_+$ and $w \in \mathcal{W}_1$, then
\[
\frac{\| s(h) \|_1 \mu_1}{1 + \gamma \| (k \ast s)(\delta) \|_1} \leq \hat{e} \leq \frac{\| s(h) \|_1 \mu_1}{1 - \gamma \| (k \ast s)(\delta) \|_1}
\] (45)
provided the nominal response $e_0(t, w)$ is bounded (i.e. the nominal closed loop system is stable) and $\gamma \| (k \ast s)(\delta) \|_1 < 1$.

**Proof.** The proof follows that of Theorem 1 except that equation (6) is used to substitute $s(\delta) \ast w(t)$ by $(s(h) \ast w)(t)$ prior to the application of Holder’s inequality, and Lemma 2 is used instead of Lemma 1.

5.2 Multiplicative uncertainty

Let $g \in \mathcal{G}_\times$ where $\mathcal{G}_\times$ represents some set of plants. Define a nominal linear time invariant plant $g_0$ such that $g_0 \in \mathcal{G}_\times$ and
\[
g = z_\times \ast g_0 + g_0
\] (46)
where $z_\times$ is termed the multiplicative uncertainty or perturbation. Define
\[
\mathcal{G}_\times := \{ g = z_\times \ast g_0 + g_0 : \| z_\times(\tau, \delta) \|_1 \leq \gamma \forall \tau \in \mathbb{R} \}.
\] (47)

**Theorem 3.** For the feedback system of Figure 1, if $g \in \mathcal{G}_\times$ and $w \in \mathcal{W}_0$, then
\[
\frac{\| s(\delta) \|_1 \mu_0}{1 + \gamma \| (g_0 \ast k \ast s)(\delta) \|_1} \leq \hat{e} \leq \frac{\| s(\delta) \|_1 \mu_0}{1 - \gamma \| (g_0 \ast k \ast s)(\delta) \|_1}
\] (48)
provided the nominal response $e_0(t, w)$ is bounded (i.e. the nominal closed loop system is stable) and $\gamma \| g_0 \ast k \ast s \|_1 < 1$.

**Proof.** Substituting (46) into (19) gives
\[
e = w - z_\times \ast g_0 \ast u - g_0 \ast u
\] (49)
\[
= w - z_\times \ast g_0 \ast u - g_0 \ast k \ast e
\] (50)
which from (27) gives
\[
e = s \ast w - z_\times \ast g_0 \ast k \ast s \ast e
\] (51)
which gives
\[
\| e \|_\infty \leq \| s \ast w \|_\infty + \| z_\times \ast g_0 \ast k \ast s \ast e \|_\infty
\] (52)
by the triangular inequality. The remainder of the proof follows the same procedure as for the proof of Theorem 1.
Theorem 4. For the feedback system of Figure 1, if \( g \in \mathcal{G}_c \) and \( w \in \mathcal{W}_1 \), then

\[
\frac{\| s(h) \|_1 \mu_1}{1 + \gamma \| (g_0 * k * s)(\delta) \|_1} \leq \hat{e} \leq \frac{\| s(h) \|_1 \mu_1}{1 - \gamma \| (g_0 * k * s)(\delta) \|_1}
\]

provided the nominal response \( e_0(t, w) \) is bounded (i.e., the nominal closed loop system is stable) and \( \gamma \| (g_0 * k * s)(\delta) \|_1 < 1 \).

Proof. The proof follows the same procedure as for the proof of Theorems 3 and 2.

5.3 Co-prime factor uncertainty

Let \( g \in \mathcal{G}_c \) where \( \mathcal{G}_c \) represents some set of plants. Define a nominal linear time invariant plant \( g_0 \) such that \( g_0 \in \mathcal{G}_c \) and denoting the Laplace transform of \( g_0 \) by \( G_0(p) \), define

\[ G_0(p) =: C(p) / D(p) \]

where \( C(p) \) and \( D(p) \) are proper, stable, transfer functions and there exist some stable \( X(p), Y(p) \) such that \( C(p)X(p) + D(p)Y(p) = 1 \), that is \( C(p), D(p) \) have no common zeros. This is the well-known co-prime factorization.

Let the plant have additive uncertainty on the co-prime factors,

\[ g = (c + z_c) * (d + z_d)^{-1}, \]

and define

\[ \mathcal{G}_c := \{ g = (c + z_c) * (d + z_d)^{-1} : \| z_c(\tau, \delta) \|_1 \leq \gamma_c, \| z_d(\tau, \delta) \|_1 \leq \gamma_d \forall \tau \in \mathbb{R} \}. \]

Theorem 5. For the feedback system of Figure 1, if \( g \in \mathcal{G}_c \) and \( w \in \mathcal{W}_0 \), then

\[
\frac{\left(\| s(\delta) \|_1 - \gamma_d \right) (s * d^{-1})(\delta)) \|_1 \mu_0}{1 + \gamma_d \| (s * d^{-1})(\delta)) \|_1 + \gamma_c \| (s * d^{-1} * k)(\delta) \|_1} \leq \hat{e} \leq \frac{\left(\| s(\delta) \|_1 + \gamma_d \right) (s * d^{-1})(\delta)) \|_1 \mu_0}{1 - \gamma_d \| (s * d^{-1})(\delta)) \|_1 + \gamma_c \| (s * d^{-1} * k)(\delta) \|_1}.
\]

provided the nominal response \( e_0(t, w) \) is bounded (i.e., the nominal closed loop system is stable) and \( \gamma_d \| (s * d^{-1})(\delta)) \|_1 + \gamma_c \| (s * d^{-1} * k)(\delta) \|_1 \| < 1 \).

Proof. Substituting (55) into (19) gives

\[
e = w - (c + z_c) * (d + z_d)^{-1} * u \]

\[
e = w - (c + z_c) * (d + z_d)^{-1} * k * e
\]

and rearranging gives

\[ d * e + c * k * e = (d + z_d) * w - z_d * e - z_c * k * e \]

and

\[ e + g_0 * k * e = w + d^{-1} * z_d * w - d^{-1} * z_d * e - d^{-1} * z_c * k * e. \]

Substituting \( s \) from (27) gives

\[
e = s * w + s * d^{-1} * z_d * w - s * d^{-1} * z_d * e - k * s * d^{-1} * z_c * e.
\]
From the triangular inequality
\[
\|e\|_\infty \leq \|s * w\|_\infty + \|s * d^{-1} * z_d * w\|_\infty + \|s * d^{-1} * z_d * e\|_\infty + \|k * s * d^{-1} * z_c * e\|_\infty. \tag{63}
\]

From Holder’s inequality
\[
\|e\|_\infty \leq \|s(\delta)\|_1 \|w\|_\infty + \|(s * d^{-1} * z_d)(\tau, \delta)\|_1 \|w\|_\infty + \|(s * d^{-1} * z_d)(\tau, \delta)\|_1 \|e\|_\infty + \|(k * s * d^{-1} * z_c)(\tau, \delta)\|_1 \|e\|_\infty. \tag{64}
\]

From the sub-multiplicative property of the $\mathcal{L}_1$-norm
\[
\|e\|_\infty \leq \|s(\delta)\|_1 \|w\|_\infty + \|(s * d^{-1})\|_1 \|z_d(\tau, \delta)\|_1 \|w\|_\infty + \|(s * d^{-1})\|_1 \|z_d(\tau, \delta)\|_1 \|e\|_\infty + \|(k * s * d^{-1})\|_1 \|z_c(\tau, \delta)\|_1 \|e\|_\infty, \tag{65}
\]
and from (56) gives
\[
\|e\|_\infty \leq \|s(\delta)\|_1 \|w\|_\infty + \gamma_d\|s * d^{-1}\|_1 \|w\|_\infty + \gamma_c\|(k * s * d^{-1})\|_1 \|e\|_\infty \forall g \in \mathcal{G}_c. \tag{66}
\]

Rearranging gives
\[
\sup_{g \in \mathcal{G}_c} \|e\|_\infty \leq \frac{\|s(\delta)\|_1 + \gamma_d\|s * d^{-1}\|_1}{1 - (\gamma_d\|s * d^{-1}\|_1 + \gamma_c\|(k * s * d^{-1})\|_1)} \tag{67}
\]
if
\[
(\gamma_d\|s * d^{-1}\|_1 + \gamma_c\|(k * s * d^{-1})\|_1) < 1. \tag{68}
\]

So from (16),
\[
\sup_{w \in \mathcal{W}} \sup_{g \in \mathcal{G}_c} \|e\|_\infty \leq \frac{\|s(\delta)\|_1 + \gamma_d\|s * d^{-1}\|_1}{1 - (\gamma_d\|s * d^{-1}\|_1 + \gamma_c\|(k * s * d^{-1})\|_1)} \mu_0 \tag{69}
\]
and the right hand inequality of (57).

Now from (62),
\[
s * w - e = s * d^{-1} * z_d * e + k * s * d^{-1} * z_c * e - s * d^{-1} * z_d * w. \tag{70}
\]
which from the triangular inequality, Holder’s inequality, the sub-multiplicative property of the $\mathcal{L}_1$-norm and from (56) gives
\[
\|s * w - e\|_\infty \leq \gamma_d\|s * d^{-1}\|_1 \|e\|_\infty + \gamma_c\|(k * s * d^{-1})\|_1 \|e\|_\infty + \gamma_d\|s * d^{-1}\|_1 \|w\|_\infty \forall g \in \mathcal{G}_c. \tag{71}
\]
From (42)
\[
\|s * w\|_\infty \leq \|e\|_\infty + \gamma_d\|s * d^{-1}\|_1 \|e\|_\infty + \gamma_c\|(k * s * d^{-1})\|_1 \|e\|_\infty + \gamma_d\|s * d^{-1}\|_1 \|w\|_\infty \forall g \in \mathcal{G}_c. \tag{72}
\]
Rearranging gives
\[
\sup_{g \in G_c} \|e\|_\infty \geq \frac{\|s \ast w\|_\infty - \gamma_d \| (s \ast d^{-1})(\delta) \|_1 \|w\|_\infty}{1 + \gamma_d \| (s \ast d^{-1})(\delta) \|_1 + \gamma_c \| (k \ast s \ast d^{-1})(\delta) \|_1},
\]
and since, from Lemma 1, \( \|s \ast w\|_\infty = \|s(\delta)\|_1 \mu_0 \) for all \( w \in W_0 \), and from Definition 3, \( \|w\|_\infty = \mu_0 \) for all \( w \in W_0 \), then
\[
\sup_{c \in W_0} \sup_{g \in G_c} \|e\|_\infty \geq \frac{\|s(\delta)\|_\infty - \gamma_d \| (s \ast d^{-1})(\delta) \|_1 \mu_0}{1 + \gamma_d \| (s \ast d^{-1})(\delta) \|_1 + \gamma_c \| (k \ast s \ast d^{-1})(\delta) \|_1},
\]
and the left-hand inequality of (57).

**Theorem 6.** For the feedback system of Figure 1, if \( g \in G_c \) and \( w \in W_1 \), then
\[
\frac{\|s(h)\|_1 - \gamma_d \| (s \ast d^{-1})(h) \|_1 \mu_1}{1 + (\gamma_d \| (s \ast d^{-1})(\delta) \|_1 + \gamma_c \| (s \ast d^{-1} * k)(\delta) \|_1)} \leq \hat{e} \leq \frac{\|s(h)\|_1 + \gamma_d \| (s \ast d^{-1})(h) \|_1 \mu_1}{1 - (\gamma_d \| (s \ast d^{-1})(\delta) \|_1 + \gamma_c \| (s \ast d^{-1} * k)(\delta) \|_1)}.
\]
provided the nominal response \( e_0(t,w) \) is bounded (i.e. the nominal closed loop system is stable) and \( (\gamma_d \| (s \ast d^{-1})(\delta) \|_1 + \gamma_c \| (s \ast d^{-1} * k)(\delta) \|_1) < 1. \)

**Proof.** The proof follows the same procedure as for the proof of Theorems 5 and 2.

6 Comments and conclusion

In this paper, some practical conditions for the system to be robustly matched to the environment have been established using a functional analysis approach. The conditions for additive uncertainty in Theorem 2 are similar to conditions established in [Zak83, Zak96]. The conditions are more conservative than that of [Zak83, Zak96], but are more readily tractable, relying on an open loop estimation of the uncertainty set. The right-hand inequalities of (30), (45), (48), (53), (57) and (75) can actually be immediately deduced from the small-gain theorem, and equivalent expressions are actually very well-known in the \( H_\infty \) setting, particularly for the multiplicative uncertainty case (e.g. [DFT91, p. 58]). Generalizations can be found in [Vid93, pp 337-343].

The co-prime uncertainty model used for Theorems 5 and 6 is well-known, (e.g. [DDB95, p. 49], [ZDG96, pp 225-226]). The advantage of this uncertainty description is that it allows perturbations that would otherwise not be stable if modelled as additive or multiplicative perturbations. It also should be noted that the control \( u \), is usually required to be bounded, practical conditions for achieving a robust match for this signal are also required.

For the practical design of most engineering control systems, it is important to explicitly consider uncertainty within the system. Hence to design the control system within the framework of matching, practical robust matching conditions are required. This paper provides matching conditions for systems with additive, multiplicative and co-prime factor uncertainty. From these matching conditions, controllers which robustly match the system to the environment can be designed using the method of inequalities [ZAN73].

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References


