Optimal Finite-precision Implementations of Linear Parameter Varying Controllers

James F Whidborne

Department of Aerospace Sciences, Cranfield University, UK

Philippe Chevrel

IRCCyN, Nantes, France

Introduction — the FWL Effects

- set of real numbers that can be stored in a digital computer is a subset of all real numbers because of Finite-Word-Length (FWL)
- constants and variables in a digital computer are subject to rounding and have a finite range (the FWL effects)

There are three main problems arising from FWL effects:

- **(i) coefficient sensitivity problem** — errors resulting from finite precision in the controller coefficients
- **(ii) round-off noise problem** — errors resulting from rounding of variables after each arithmetic computation
- **(iii) overflow/underflow problem** — limitations imposed by the finite range of variables and constants (also the *scaling* problem)

We consider coefficient sensitivity problem for LPV digital controllers

Often important to reduce wordlength to reduce controller complexity (Roger Brockett plenary)

There appears to be little previous work on LPV controller implementation:

- Apkarian (1997) considers discretization problem
- Kelly and Evers (1997) recommends balanced realizations for gain-scheduling problems
- ‘resilience’ problem for periodically varying linear state feedback controllers has been studied by Farges et al. (2007)

FWL problems for LPV controllers seem not to have been studied

Coefficient Sensitivity Problem

FWL effects are strongly dependent on the controller realization

LPV state-space controller has form

$$\begin{aligned}x(k+1) &= A(\theta(k))x(k) + B(\theta(k))y(k) \\ u(k) &= C(\theta(k))x(k) + D(\theta(k))y(k).\end{aligned}$$

All equivalent state-space realizations are given by

$$\begin{aligned}\tilde{x}(k+1) &= T^{-1}A(\theta(k))T\tilde{x}(k) + T^{-1}B(\theta(k))y(k) \\ u(k) &= C(\theta(k))T\tilde{x}(k) + D(\theta(k))y(k)\end{aligned}$$

where T is non-singular

Problem: determine T such that closed-loop LPV system is insensitive to rounding in controller coefficients

Discrete-time LPV plant

$$x_p(k+1) = A_p(\theta(k))x_p(k) + B_p u(k)$$

$$y(k) = C_p x_p(k)$$

where A_p depends affinely on the time-varying parameter vector, $\theta(k)$, and $\theta(k)$ is known at the sample instant, k (i.e. the measurement is available in real time)

Assume LPV controller

$$x(k+1) = A(\theta(k))x(k) + B(\theta(k))y(k)$$

$$u(k) = C(\theta(k))x(k) + D(\theta(k))y(k).$$

has been designed, and where $R(\theta(k))$ depends affinely on θ where

$$R := \begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix}$$

Closed loop system matrix is given by

$$A_c = \begin{bmatrix} A(\theta(k)) & B(\theta(k))C_p \\ B_p C(\theta(k)) & A_p(\theta) + B_p D(\theta(k))C_p \end{bmatrix}$$

Defining

$$A_0 := \begin{bmatrix} 0 & 0 \\ 0 & A_p \end{bmatrix} \quad B_I := \begin{bmatrix} I & 0 \\ 0 & B_p \end{bmatrix} \quad C_I := \begin{bmatrix} I & 0 \\ 0 & C_p \end{bmatrix}$$

we get

$$A_c = A_0(\theta(k)) + B_I R(\theta(k)) C_I,$$

which is also affinely dependent on θ

Due to rounding of coefficients, controller matrix R is perturbed to $R + \Delta$ and closed loop system matrix is perturbed to $A_c + B_I \Delta C_I$
Let maximum perturbation be given by the max norm

$$\|\Delta\|_{\max} := \max_{i,j} |\Delta_{i,j}|$$

and define the FWL stability margin as

$$\eta_0 := \inf \{ \|\Delta\|_{\max} : A_c + B_I \Delta C_I \text{ is unstable} \}$$

η_0 is hard to compute

Fialho and Georgiou (1994) propose using spectral norm

$$\|\Delta\|_2 := \max \left\{ \sqrt{\lambda_i} : \lambda_i \text{ are the eigenvalues of } \Delta^T \Delta \right\}$$

with FWL stability margin given by complex stability radius

$$\eta_c := \inf \left\{ \|\Delta\|_2 : A_C + B_I \Delta C_I \text{ is unstable} \right\}$$

which can be easily computed by

$$\eta_c = \frac{1}{\|C_I(zI - A_c)^{-1}B_I\|_\infty}$$

and $\|\cdot\|_\infty$ denotes the $\mathcal{H}_\infty$ -norm

Since $\|\Delta\|_{\max} \leq \|\Delta\|_2$, then η_c provides an upper bound on η_0

Problem: find T that maximizes η_c
Equivalent to the $\mathcal{H}_\infty$ minimization problem

$$\min_{T \text{ non singular}} \| C_I(zI - A_T(T))^{-1} B_I \|_\infty$$

where

$$A_T(T) := \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} A_c \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}$$

Can be solved by solving a sequence of LMI problems as proposed by Fialho and Georgiou (2001)

Following Apkarian et al. (1995b), we will consider polytopic LPV systems
Define a matrix polytope as the convex hull of r matrices, $N_1, N_2, \dots, N_r$,

$$\text{Co}\{N_i, i = 1, \dots, r\} := \left\{ \sum_{i=1}^r \alpha_i N_i : \alpha_i \geq 0, \sum_{i=1}^r \alpha_i = 1 \right\}.$$

We assume that the discrete time varying parameter, $\theta(k)$, is confined to the the polytope, Θ ,
with vertices $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_r$, that is

$$\theta(k) \in \Theta,$$

where

$$\Theta := \text{Co}\{\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_r\}$$

and that the dependence of the state space matrices on θ is affine

A polytopic system has quadratic $\mathcal{H}_\infty$ performance (Apkarian et al., 1995b) of γ if and only if there exists a Lyapunov function $V(x) = x^T P x$ with $X > 0$ that establishes global stability and ensures that the $\mathcal{L}_2$ gain of the system is bounded by γ . That is $\|y\|_2 < \gamma \|u\|_2$ along all possible parameter trajectories $\theta(k) \in \Theta$.

The closed loop LPV system is affine in θ and is hence a polytopic LPV system

So if we replace the stability radius maximization problem of the LTI case by a quadratic $\mathcal{H}_\infty$ performance (these are equivalent for LTI system) we can solve a minimal coefficient sensitivity problem for the LPV system

So we just need to solve a system of LMIs that minimizes the $\mathcal{H}_\infty$ performance measure at each vertex— thus the following is proposed

Proposition

The optimal quadratic $\mathcal{H}_\infty$ performance, γ_{opt} is the minimum γ for which there exists a $P = P^T > 0$ of the form

$$P = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & I \end{bmatrix}$$

such that

$$M_i^T(\gamma) P M_i(\gamma) < P, \text{ for } i = 1, 2, \dots, r$$

where

$$M_i(\gamma) := \begin{bmatrix} A_c(\hat{\theta}_i) & B_I/\gamma \\ C_I & 0 \end{bmatrix}$$

The optimal nonsingular transformation matrix is obtained from $P_2 = T_{\text{opt}}^T T_{\text{opt}}$

Example

Continuous-time state space plant

$$\dot{x} = A_g(\alpha_1, \alpha_2)x + B_g u,$$

$$y = C_g x$$

$$\alpha_1 + \alpha_2 = 1, \alpha_1 \geq 0, \alpha_2 \geq 0,$$

with

$$A_g = \begin{bmatrix} -1/100 & 0 & 0 \\ 0 & 0 & 1 \\ (0.2\alpha_1 + 2\alpha_2) & -(0.2\alpha_1 + 2\alpha_2) & -(0.1\alpha_1 + \alpha_2) \end{bmatrix} \quad B_g = \begin{bmatrix} 1/100 \\ 0 \\ 0 \end{bmatrix} \quad C_g = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

Weighting functions

$$W_1(s) = \frac{(s + 1/5)}{1.8(s + 1/5000)}$$

$$\text{and } W_2(s) = \frac{(s/50 + 1)}{(s/10000 + 10)}$$

The MATLAB LMI Toolbox function, `hinfgs`, with SKS $\mathcal{H}_\infty$ -criterion is used to obtain an LPV controller

Example

Controller at each vertex is discretized using the Tustin transformation with a sampling rate of 500Hz

The LMI

$$M_i^T(\gamma)PM_i(\gamma) < P, \text{ for } i = 1, 2$$

is repeatedly solved with a bisection search to obtain $\gamma_{\text{opt}} = 2.736 \times 10^3$

For comparison, modal and balanced gramian realizations also calculated

Frozen- α_1 $\mathcal{H}_\infty$ -norm against α_1 — optimum quadratic performance γ_{opt} is shown as the dashed line

Complex stability radius, η_c , against α_1 for frozen α_1

Eigenvalue sensitivity

A measure of the closed-loop poles sensitivity LTI systems is

$$\Psi = \sum_{k=1}^{n+m} \frac{1}{1 - |\lambda_k|} \Psi_k$$

where $\{\lambda_i : i = 1, \dots, m + n\}$ represents the set of unique closed-loop poles/ eigenvalues and

$$\Psi_k = \sum_{i=1}^{n_x} \left(\frac{\partial \lambda_k}{\partial x_i} \right)^2$$

and $\{x_i : i = 1, \dots, n_x\}$ are the controller parameters

Complex stability radius, η_c , against α_1 for frozen α_1 — note singularity at $\alpha_1 \simeq 0.634$

Conclusions

- Singularity at $\alpha_1 \simeq 0.634$ results from closed loop eigenvalues branching from real to complex pair
Problem does not arise in practise for LTI control design — pole multiplicities avoided
- Hence eigenvalue sensitivity is not generally suitable for LPV systems coefficient sensitivity
- LMI problems are easily solved — hence quadratic $\mathcal{H}_\infty$ measure suitable for LPV systems

Further work

- harder test case
- observer-controller structures
- effect of rounding on the scheduling parameter, θ
- closed loop transfer function sensitivity
- the round-off noise problem and the scaling problem

Acknowledgements to Ecole Centrale Nantes for supporting this work

References

- P. Apkarian. On the discretization of LMI-synthesized linear parameter-varying controllers. *Automatica*, 33(4):655–661, 1997.
- P. Apkarian, J. M. Biannic, and P. Gahinet. Self-scheduled H_∞ control of missile via linear matrix inequalities. *J. Guid. Control Dynam.*, 18(3):532–538, 1995a.
- P. Apkarian, P. Gahinet, and G. Becker. Self-scheduled $\mathcal{H}_\infty$ control of linear parameter-varying systems: a design example. *Automatica*, 31(9):1251–1261, 1995b.
- S. Boyd, V. Balakrishnan, E. Feron, and L. El Ghaoui. Control system analysis and synthesis via linear matrix inequalities. In *Proc. 1993 Amer. Contr. Conf.*, pages 2147–2153, 1993.
- S. Chen, J. Wu, and G. Li. Two approaches based on pole sensitivity and stability radius measures for finite precision digital controller realizations. *Syst. Control Lett.*, 45(4):321–329, April 2002.
- Z.W. Fang, J.E. Carletta, and R.J. Veillette. A methodology for FPGA-based control implementation. *IEEE Trans. Control Syst. Technology*, 13(6):977–987, November 2005.

- C. Farges, D. Peaucelle, A. Théron, and D. Arzelier. Resilient structured periodic $\mathcal{H}_2$ synthesis for a spacecraft in elliptical orbits. In *17th IFAC Symposium on Automatic Control in Aerospace*, 2007. CDROM.
- I.J. Fialho and T.T. Georgiou. Computational algorithms for sparse optimal digital controller realisations. In R.S.H. Istepanian and J.F. Whidborne, editors, *Digital Controller Implementation and Fragility: A Modern Perspective*, chapter 7, pages 105–121. Springer-Verlag, London, UK, 2001.
- I.J. Fialho and T.T. Georgiou. On stability and performance of sampled-data systems subject to word-length constraint. *IEEE Trans. Autom. Control*, 39(12):2476–2481, 1994.
- I.J. Fialho and T.T. Georgiou. Optimal finite wordlength digital controller realizations. In *Proc. 1999 Amer. Contr. Conf.*, pages 4326–4327, San Diego, CA, 1999.
- P. Gahinet, A. Nemirovski, A.J. Laub, and M. Chilali. *LMI Control Toolbox: User's Guide, version 1*. The MathWorks, Inc., Natick, MA, 1995.
- M. Gevers and G. Li. *Parametrizations in Control, Estimations and Filtering Problems: Accuracy Aspects*. Springer-Verlag, Berlin, 1993.

- T. Hilaire, P. Chevrel, and J.P. Clauzel. Pole sensitivity stability related measure of FWL realization with the implicit state-space formalism. In *Proc. 5th IFAC Symposium on Robust Control Design (ROCOND'06)*, Toulouse, France, July 2006.
- T. Hilaire, P. Chevrel, and J.F. Whidborne. A unifying framework for finite wordlength realizations. *IEEE Trans. Circuits Syst. I, Reg. Papers*, 54(8):1765 – 1774, August 2007.
- D. Hinrichsen and A.J. Pritchard. Stability radius for structural perturbations and the algebraic Riccati equation. *Syst. Control Lett.*, 8:105–113, 1986.
- R.H. Istepanian, G. Li, J. Wu, and J. Chu. Analysis of sensitivity measures of finite-precision digital controller structures with closed-loop stability bounds. *IEE Proc. Control Theory and Appl.*, 145(5):472–478, 1998.
- R.S.H. Istepanian and J.F. Whidborne, editors. *Digital Controller Implementation and Fragility: A Modern Perspective*. Springer-Verlag, London, UK, September 2001. ISBN 1852333901.
- J.H. Kelly and J.H. Evers. An interpolation strategy for scheduling dynamic compensators. In *Proc. AIAA Guidance, Navigation and Control Conf.*, pages 1682–1690, New Orleans, LA, 1997.

- D. J. Leith and W. E. Leithead. Survey of gain-scheduling analysis and design. *Int. J. Control*, 73(11):1001–1025, July 2000.
- G. Li. On the structure of digital controllers with finite word length consideration. *IEEE Trans. Autom. Control*, 43(5): 689–693, 1998.
- W.J. Rugh and J.S. Shamma. Research on gain scheduling. *Automatica*, 36(10):1401–1425, October 2000.
- J.F. Whidborne, J. Wu, R.H. Istepanian, and J. Chu. Comments on “On the structure of digital controllers with finite word length consideration”. *IEEE Trans. Autom. Control*, 45 (2):344, 2000.
- J.F. Whidborne, R.S.H. Istepanian, and J. Wu. Reduction of controller fragility by pole sensitivity minimization. *IEEE Trans. Autom. Control*, 46(2):320–325, 2001.
- J. Wu, S. Chen, G. Li, and J. Chu. Optimal finite-precision state-estimate feedback controller realizations of discrete-time systems. *IEEE Trans. Autom. Control*, 45(8):1550 – 1554, 2000a.
- J. Wu, S. Chen, G. Li, R.H. Istepanian, and J. Chu. Shift and delta operator realisations for digital controllers with finite word length considerations. *IEE Proc.-Control Theory Appl.*, 147(6):664–672, 2000b.

- J. Wu, S. Chen, G. Li, R.H. Istepanian, and J. Chu. An improved closed-loop stability related measure for finite-precision digital controller realizations. *IEEE Trans. Autom. Control*, 46(7): 1662–1666, 2001.
- W.-S. Yu and H.-J. Ko. Improved eigenvalue sensitivity for finite-precision digital controller realisations via orthogonal Hermitian transform. *IEE Proc. Control Theory and Appl.*, 150(4): 365–375, 2003.