Integration of $\mathcal{H}_\infty$ and Time Domain Specifications using the Convex Combination Method

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Introduction

Control system design problems are usually multi-objective — there are several, usually competing, requirements that need to be satisfied in order for the design to be deemed successful. For flight control systems, formulating the problem as a multiobjective one is particularly apt, as the consequences of failing to get the necessary trade-off between the competing objectives of performance, robustness, actuator effort, controller simplicity etc can be very costly. Consequently, there have been many multiobjective design approaches proposed for flight control, (Tabak *et al.*, 1979; Grübel *et al.*, 1993; Hughes *et al.*, 2002; Jeanneau *et al.*, 2005, for example). An important development in multiobjective control design was the realization that many performance indices are convex in nature (Polak and Salcudean, 1989; Boyd *et al.*, 1990). Furthermore, using the noted Q -parametrization (or Youla parametrization), the performance indices are convex with respect to the free parameter Q . Examples of flight control design using convex optimization include Cifdalo *et al.* (2003), Puyou *et al.* (2004) and Ferreres and Puyou (2006). The convex combination method was proposed by Liu and Mills (1998a), (see also Liu and Mills, 1998b). This is a simple two step design strategy for controller design that can satisfy multiple simultaneous specifications, provided the specifications are convex. In the first step, n controllers for n convex specifications are designed such that each controller satisfies at least one specification. In the second step, a linear programming problem is formulated such that some combination of the n controllers simultaneously satisfies all n specifications. If a solution exists, the final controller can be calculated. The method has been applied to flight control system design (Liu and Mills, 2000; Liu, 2001a,b; Fung *et al.*, 2005). However, the method as proposed does not utilize the Q -parametrization, and subsequently there is no guarantee that the final controller will be internally stable (Wong *et al.*, 2005). Hence this paper proposes using the Q -parametrization with the convex combination method, and provides some guidance on how this may be done.

Use of $\mathcal{H}_\infty$ design techniques is common in control design problems for objectives like robust stability and good performance in the presence of uncertainties. However such controllers are designed in the frequency domain. In some applications like aircraft control, the time domain response is also very important, e.g., for good aircraft handling qualities. These handling qualities normally put some specific limits to different time domain signals like pitch attitude, flight path angle and pitch rate etc (Gibson criteria). These additional constraints cannot be directly addressed in $\mathcal{H}_\infty$ design using the common methods, e.g. loop shaping, mixed sensitivity, etc. The convex combination method can provide a solution to such problems provided the problem can be formulated into a structure which satisfies necessary requirements for the application of the said method. In the next section, the convex combination method is briefly summarized. The following section describes how the Q -parametrization can be used with the method. A flight control example illustrates the procedure. Finally, the conclusions give some indication on how the method can be extended and generalized.

Figure 1. Required 2-port structure for the convex combination method

Aircraft flight control systems often require the satisfaction of multiple simultaneous specifications to fulfil different robust stability and performance objectives along with good handling qualities. Many specifications such as $\mathcal{H}_\infty$ -norm, overshoot, undershoot, etc are convex with respect to the closed loop transfer matrix. In this paper it is suggested to use the convex combination method to integrate $\mathcal{H}_\infty$ frequency domain specifications with additional time domain constraints. For this purpose, existing techniques are used to design an $\mathcal{H}_\infty$ controller which is then modified in a new structure using Q -parametrization which makes the problem convex in the new design parameter Q . At this stage, if different controllers are obtained for each convex specification, they can be combined into a single controller using convex combination which could satisfy all specifications.

Convex Combination Method

The objective is to design a controller $F(s)$ such that

$$\phi_i(F) \leq \psi_i, \text{ for } i = 1, \dots, n \quad (1)$$

is satisfied where ϕ_i is the i th performance index and ψ_i represents the i th specification, that being a maximum permissible value for ϕ_i .

The procedure is as follows (Liu and Mills, 1998a)

Stage 1 Design n controllers, F_i , $i = 1, \dots, n$, so that each controller satisfies

$$\phi_i(F_i) \leq \psi_i. \quad (2)$$

Stage 2 Formulate the linear programming problem

$$\Phi \cdot \Lambda \leq \Psi \quad (3)$$

where $\Phi = [\phi_i(F_j)]$, $i = 1, \dots, n$, $j = 1, \dots, n$, is a matrix of performance indices and $\phi_i(F_j)$ is the i th performance with the j th controller, $\Psi = [\psi_1, \dots, \psi_n]^T$ is the vector of design specifications and $\Lambda = [\lambda_1, \dots, \lambda_n]^T$, $\lambda_i \geq 0$, $\sum_{i=1}^n \lambda_i = 1$ is the combination vector of the free parameters. Now if a solution to the problem exists, then using the solution combination vector Λ , a final controller can be determined that simultaneously satisfies all the specifications.

Before applying the convex combination method, the system should be formulated into the well-known general configuration which separates inputs and outputs into two groups. One group contains signals important from the specification point of view while the other group contains signals which are directly given to controller. The advantage of this structure is that a single closed loop transfer matrix is obtained which could be used for all design requirements. The procedure is standard for many type of problems and the required configuration shown in Figure 1, where $P(s)$ is given as

$$P(s) = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix} \stackrel{s}{=} \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix} \quad (4)$$

where (A, B_u, C_y) is assumed stabilizable and detectable. The closed loop transfer matrix between z and w is given as

$$H_{zw} = P_{zw} + P_{zu}F(I - P_{yu}F)^{-1}P_{yw} \quad (5)$$

$$=: LFT_l(P, F) \quad (6)$$

where $LFT_l(\cdot, \cdot)$ is the well-known lower fractional transform operation.

For the convex combination method, the final controller, K^* , is obtained from the convex combined closed-loop transfer function matrix H^* where $H^* = \lambda_1 H_1 + \dots + \lambda_n H_n$ and $H_1 = LFT_l(P, F_1)$ etc. The final controller is then obtained from (5). Note that this is not always straightforward. Consequently, many of the examples of the method make use of some particular structure, e.g. open-loop (Liu and Mills, 2000), or with some special coupled structures (Liu, 2001a; de Ruiter and Liu, 2004), that simplifies this step. More importantly, there is no guarantee that the resulting controller is internally stable, and an example of this is given by Wong *et al.* (2005).

To overcome this problem, the well-known Q -parametrization can be utilized (e.g. Boyd and Barratt, 1991). The procedure for the convex combination method is summarized next.

Q -parametrization

It is evident from (5) that the controller $F(s)$ appears in a nonlinear way in the closed loop transfer matrix H_{zw} and is thus not convex in $F(s)$. To achieve convexity, Q -parametrization (Boyd and Barratt, 1991) is used.

Figure 2. Q -Parametrization

To start with, it is assumed that the open loop plant $P(s)$ is unstable. Now with this plant, a stabilizing nominal controller K_{nom} is added. The primary objective of K_{nom} is to stabilize the plant and to ensure that the transfer function H_{ev} between e and v is zero to make the problem convex. However at this stage it is not ensured that H_{ev} is zero. Now if we replace the extended output e to $\tilde{e}$ and attach a new transfer function $\tilde{Q}$ then the controller $F(s)$ is given as

$$F(s) = LFT_l(K_{nom}, \tilde{Q}) \quad (7)$$

In the next step we will transform both $\tilde{Q}$ and $\tilde{e}$ into Q and e which will ensure that H_{ev} is zero. For this purpose, first, the plant and nominal controller are combined into a new transfer matrix $\tilde{T}$ as shown in Figure 2 while $\tilde{T}$ is given as

$$\tilde{T} = LFT_l(P, K_{nom}) =: \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix} \quad (8)$$

Figure 3. Convex Q -Parametrization

Now, to make the problem convex, a further transformation is performed, as shown in Figure 3. This transformation ensures that H_{ev} is zero. It is now simple to represent the upper dotted block with T which is given as

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & 0 \end{bmatrix} \quad (9)$$

Similarly we can replace the lower dotted block of Figure 3 with Q which is represented as

$$Q = (I - \tilde{Q}T_4)^{-1}\tilde{Q} = \tilde{Q}(I - T_4\tilde{Q})^{-1} \quad (10)$$

The final structure achieved after all of these transformations is an LFT, $H_{zw} = LFT_l(T, Q)$, and because T_4 is zero, the closed loop transfer matrix is

$$H_{zw} = T_1 + T_2QT_3 \quad (11)$$

and is convex in Q .

Now for different specifications, different Q transfer functions can be designed remaining within the transformed framework. In such a case, the final satisfactory transfer matrix can be achieved with the help of following equation

$$H_{zw}^* = \sum_{i=1}^n H_{zw}^{(i)} \quad (12)$$

$$= T_1 + T_2 \left(\sum_{i=1}^n \lambda_i Q_i \right) T_3 \quad (13)$$

$$= T_1 + T_2 Q^* T_3 \quad (14)$$

Here λ_i is the solution of a linear programming problem. Once a satisfactory Q^* is designed, the final controller can be obtained using (7) and (10) which will satisfy all specifications provided a solution to the linear programming problem, (3), exists.

In the previous discussion, it was assumed that a stabilizing controller K_{nom} was required. However if the plant is stable, K_{nom} can be zero and the algebra is considerably simplified. To fulfil the convexity condition and make the transfer function H_{ev} equal to zero, we set $T_4 = P_{yu}$. Hence Q includes both $\tilde{Q}$ and P_{yu} and the required $T(s)$ can be achieved as given by (9) with T_1 , T_2 and T_3 replaced by P_{zw} , P_{zu} and P_{yw} . In this case, the final closed loop transfer matrix is given as

$$H_{zw} = P_{zw} + P_{zu}QP_{yw} \quad (15)$$

and Q by

$$Q = F(I - P_{yu}F)^{-1} \quad (16)$$

where F is the controller. In a similar way to (14), we can find a satisfactory Q^* for different specifications. The final controller can be obtained using

$$F = (I + Q^*P_{yu})^{-1}Q^* \quad (17)$$

Thus using these basic concepts and relations, the convex combination method can be applied on variety of control design problems.

Case Study

The HiMAT Research Aircraft

The presented method is applied to a HiMAT aircraft linear longitudinal model. HiMAT (Highly Maneuverable Aircraft Technology) was a research aircraft developed and used by NASA in 1980s. The HiMAT (Highly Maneuverable Aircraft Technology) subscale research vehicle, seen here during a research flight, was flown by the NASA Dryden Flight Research Center, Edwards, California, from mid 1979 to January 1983. The aircraft demonstrated advanced fighter technologies that have been used in the development of many modern high performance military aircraft. A linear model was obtained by trimming the aircraft at 25000 feet altitude and 0.9 Mach speed (Cifdaloz *et al.*, 2003). This linear model is a MIMO model having an unstable phugoid mode. The inputs are the angles of the elevon and canard and the outputs are the angle of attack, α , and the pitch angle θ .

Design Specifications

1. Mixed sensitivity $\mathcal{H}_\infty$ specifications

Mixed sensitivity $\mathcal{H}_\infty$ specifications were set for the sensitivity function, S , and complementary sensitivity function, T . More specifically, these specifications can be given as:

- Performance Specification: Minimize the sensitivity function as much as possible.
- Robustness Specification: -40 dB/decade roll off and at least -20 dB at 100 rad/sec.

These specification can be collectively represented as:

$$\left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty \leq \gamma \quad (18)$$

where W_1 and W_2 are the weights which provide a reference singular value response according to defined specifications. Appropriate values for the weights were also taken from Cifdaloz *et al.* (2003).

2. Time domain specification

The following step response specification is included:

- Peak overshoot of the pitch angle step response, $\theta(t)$, should be less than 20 %.

Design Process

To solve the problem, initially an $\mathcal{H}_\infty$ controller was designed by solving (18) using the MATLAB Robust Control Toolbox. Then the problem was formulated into the two-port structure of Figure 1. The $\mathcal{H}_\infty$ controller problem is formulated as in Figure 4. Here the signals that are important from the specifications point of view are included in transfer matrix H_{zw} where $z = [z_\infty, z_t]^T$ where $z_\infty = [W_1 S, W_2 T]^T w$ and $z_t = [0, 1]^T \theta$.

From the controller obtained by solving specification (1), a nominal controller K_{nom} is extracted. The controller designed is represented as

$$F_1(s) = LFT_1(K_{nom}, 0) \quad (19)$$

that is $Q_1(s) = 0$. The step response, $\theta(t)$, has a peak overshoot of 38% which does not satisfy specification (2).

Once K_{nom} is calculated, $T(s)$ of the form of (9) is easily obtained. Apart from fulfilling the convexity requirement, the Q parametrization also has the property that $Q(s)$ cannot make the system unstable provided that $Q(s)$ is stable. This is because the stability has already been provided through the nominal controller, and internal stability is guaranteed (Boyd and Barratt, 1991). However the designer can change $Q(s)$ to affect the time-domain responses of different signals.

Designing a suitable $Q_2(s)$ that meets specification (2) is the next task. At this stage, $Q(s)$ could be parameterized using a Ritz approximation Boyd and Barratt (1991), and convex optimization algorithms used to find a $Q_2(s) = Q(s)$ that satisfies specification (2) (or indeed simultaneously all the specifications). However for this example, $Q_2(s)$ was obtained by manual iterations on a second order transfer function which could provide improved time domain specifications. Hence we have $F_2 = LFT_1(K_{nom}, Q_2)$. The performance specifications for controllers F_1 and F_2 are given in the table.

Convex Combination

Based on the specification values given in the table below, the linear programming problem, (3), was formulated as

$$\begin{bmatrix} 0.217 & 0.250 \\ 38.59 & 16.93 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \leq \begin{bmatrix} 0.220 \\ 20 \end{bmatrix} \quad (20)$$

The above feasibility problem was solved and a combination vector was obtained as

$$\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0.141 \\ 0.859 \end{bmatrix} \quad (21)$$

Now using equation (14), (10) and (7), the final controller $F^*(s)$ was obtained.

Results

Comparison of specification values — the final controller provided a solution to the problem, achieving best of both first step design specification values.

Specifications	F_1	F_2	Ψ	F^*
$\Phi_{\mathcal{H}_\infty}$	0.217	0.250	0.220	0.215
Φ_{OS}^θ	38.6 %	16.9%	20%	19.9%

Step response with different controllers F_1 , F_2 and F^ — the peak overshoot of the pitch attitude step response, $\theta(t)$, has been reduced from 38% to around 20%*

Sensitivity function of final controller, $F^(s)$ — the bounds set by weight W_1 are still satisfied*

Complementary sensitivity function of final controller, $F^(s)$ — the bounds set by weight W_2 are still satisfied*

Conclusions

A technique to combine different $\mathcal{H}_\infty$ controllers was presented in order to integrate frequency domain $\mathcal{H}_\infty$ specifications with time domain specifications. The idea was applied on a test case and found workable.

The usual methods for convex optimization using the Q -parametrization require a parametrization of the free parameter transfer function matrix Q . Choosing an appropriate basis function for this can be hard (Linnemann, 1999). Furthermore, the order of the problem can grow very quickly, and the search can become slow (Whidborne and McKernan, 2007). The advantage of the convex optimization method is that it is not required to parameterize Q , and requires only the solution of a simple linear programming problem.

A limitation observed in the convex combination method is that the resulting controllers are high order. For the case study, the plant is 6th order, but the final combined controller is 16th order. Controller order reduction techniques have been implemented on this problem with satisfactory results (Ahmed, 2004). For example, using a balanced truncation method, the controller order can be reduced to 9 while still satisfying the specifications (Ahmed, 2004).

The approach suggested in this paper is similar to that of de Ruiter and Liu (2003) in that it uses a Q -parametrization for the convex combination method. In this study, the free parameter, $Q(s)$, was tuned manually to obtain a satisfactory step response. This approach is not entirely satisfactory, because we would like to be able to design the controllers F_i , $i = 1, \dots, n$, by any method. The problem that needs to be addressed is:

Problem Given n controllers $F_i(s)$, $i = 1, \dots, n$, each controller satisfying at least one performance criterion, the problem is to obtain a convex combination of these controllers. That is, if $F_i = LFT_1(K_i, Q_i)$, determine $\hat{Q}_i$, $i = 2, \dots, n$ such that $F_i = LFT_1(K_1, \hat{Q}_i)$.

As far as we are aware, this is an open problem and remains for future work.

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